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PHYSICS

ON CERTAIN FINITE DIFFERENCE SCHEMES FOR THE EQUATIONS OF HYDRODYNAMICS

by

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March 1, 1962

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ABSTRACT

We are concerned with the application of two finite difference schemes of second order accuracy to the equations of viscid and inviscid flow. The first is the Lax-Wendroff scheme, the second an iterative scheme. A stability criterion is obtained for the iterative scheme by the method of von Neumann. An empirical stability criterion is obtained for the Lax-Wendroff scheme as applied to viscid flow. The accuracy of these methods and their effectiveness for flows which contain a shock are also discussed.

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ON CERTAIN FINITE DIFFERENCE SCHEMES FOR THE EQUATIONS OF HYDRODYNAMICS

1. Introduction.

We will consider two finite difference schemes for the solution of the Navier-Stokes equations in one space dimension. The first is a method used by Lax and Wendroff [1]. The second method is based on an approximate solution of the centered, implicit difference equations. Since this approximate solution is obtained by successive substitutions, we will refer to this second method as an iterative method.

By using the method of von Neumann we will obtain a stability criterion for the iterative method as applied to the equations for inviscid flow. This method is stable if $3, 4, 7, 8, \ldots$ iterations are used per time-step and unstable if $1, 2, 5, 6, \ldots$ iterations are used. We will obtain an empirical stability criterion for the Lax-Wendroff method as applied to the Navier-Stokes equations.

Both of these methods have a truncation error of second order. We will describe the results of calculations in which both of these methods were applied to a problem for which an exact solution is known. Thus we will obtain an indication of the accuracy of these methods.

P. Lax has applied the Lax-Wendroff method, as well as a method of first order accuracy, to inviscid flows which

contain a shock of moderate strength [1,2]. He used the equations of motion in conservation form in order to handle the discontinuity without the introduction of artificial viscosity or the use of shock fitting. We will compare the results of computing with the iteration method and the Lax-Wendroff method for inviscid flows which contain a shock. It will be shown that the iteration method with the equations in conservation form does not work as well as the Lax-Wendroff method where the equations are not in conservation form.

2. The Finite Difference Equations.

We let ρ, p, u, e, m, γ, and R denote density, pressure, velocity, energy, momentum, ratio of specific heats, and Reynolds number ($m = \rho u$ and $e = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}$). The Navier-Stokes equations can be written as

$$\frac{\partial v}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} = 0$$

where

$$v = \begin{bmatrix} \rho \\ e \\ m \end{bmatrix} \qquad f(v) = \begin{bmatrix} m \\ \frac{\gamma m e}{\rho} - \frac{\gamma-1}{2}\frac{m^3}{\rho^2} \\ (\gamma-1)e - \frac{\gamma-3}{2}\frac{m^2}{\rho} \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ -\frac{4}{3R}\frac{m}{\rho}\frac{\partial(\frac{m}{\rho})}{\partial x} \\ -\frac{4}{3R}\frac{\partial(\frac{m}{\rho})}{\partial x} \end{bmatrix}$$

We denote the forward and backward differences by $v_x = v(t,x+\Delta x) - v(t,x)$ and $v_{\bar{x}} = v(t,x) - v(t,x-\Delta x)$ and the centered difference by $v_{\hat{x}} = {}^1/2(v_{\bar{x}} + v_x)$. We use the notation $v_i^n = v(t_n, x_i)$ and denote the Jacobian of

$f(v)$ with respect to v by A. The Lax-Wendroff method is based on the expansion

$$(2.1)\quad \left(\frac{\partial v}{\partial t}\right)_i^n = -\left(\frac{\partial f}{\partial x}\right)_i^n - \left(\frac{\partial g}{\partial x}\right)_i^n - \frac{\Delta t}{2}\left[\left(\frac{\partial^2 f}{\partial x \partial t}\right)_i^n + \left(\frac{\partial^2 g}{\partial x \partial t}\right)_i^n\right] ,$$

(note that $(\frac{\partial f}{\partial x})_i^n = (A\,\frac{\partial v}{\partial x})_i^n$). In order to obtain a difference approximation for the term $\partial^2 g/\partial x \partial t$ we need a difference approximation for $\partial u/\partial t$. This is obtained from the Navier-Stokes equation

$$\frac{\partial u}{\partial t} = -u\,\frac{\partial u}{\partial x} - \frac{1}{\rho}\,\frac{\partial p}{\partial x} + \frac{4}{3R\rho}\,\frac{\partial^2 u}{\partial x^2}$$

by replacing derivatives by difference quotients. If we denote this approximation by $\Delta_t u$ and denote the similar approximation for $\partial g/\partial t$ by $\Delta_t g$, then

$$\Delta_t u = -uu_x - \frac{1}{\rho}\,p_x + \frac{4}{3R\rho}\,u_{x\bar{x}} \quad \text{and}$$

$$(\Delta_t g)^* = \left(0,\ -\frac{4}{3R}\,(u(\Delta_t u)_x + (\Delta_t u)u_x,\ -\frac{4}{3R}\,(\Delta_t u)_x\right).$$ (g^* denotes the transpose of g)

The Lax-Wendroff method is obtained by replacing derivatives with difference quotients. The difference equations are

$$v^{n+1} = v^n - \lambda\left(f_{\hat{x}}^n + g_{\hat{x}}^n\right) - \frac{\lambda^2}{2}\left[(A\Delta v_t)_{\bar{x}}^n + (\Delta_t g)_{\bar{x}}^n\right]$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\Delta v_t = -f_x - \tilde{g}_x$, and $\tilde{g}^* = (0,\ -\frac{4}{3R}\,uu_x,\ -\frac{4}{3R}\,u_x)$.

In order that the truncation error be $O(\Delta x^3)$ the differences $f_{\hat{x}}$ and $g_{\hat{x}}$ must be centered. Since the last term on the right is of second order, the differences in this term need not be centered. In the calculations the boundary values of v, that is v_1^n and v_M^n, are known and the values of v_i^n for $i = 2,3,\ldots M-1$ and $n = 2,3,\ldots$ are computed. Since the second order term in the difference equations requires the values v_{i-2}^n, v_{i-1}^n, v_i^n, v_{i+1}^n, and v_{i+2}^n to compute v_i^{n+1}, we must modify this term when $i = 2$ or $i = M-1$. The modification merely requires that certain forward differences be replaced by backward differences, or conversely.

We also performed calculations using the Lax-Wendroff method for the Navier-Stokes equations where the latter are not written in conservation form. In this case the equations are

$$\frac{\partial \rho}{\partial t} = - \rho\frac{\partial u}{\partial x} - u\frac{\partial \rho}{\partial x}$$

$$(2.2) \quad \frac{\partial p}{\partial t} = - u\frac{\partial p}{\partial x} - \gamma p\frac{\partial u}{\partial x} + \frac{4(\gamma-1)}{3R}\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial u}{\partial t} = - u\frac{\partial u}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{4}{3R\rho}\frac{\partial^2 u}{\partial x^2} .$$

The difference equations are obtained in the same manner as before.

The iteration method is based on an implicit form of the difference equations. These are

$$v^{n+1} = v^n - \frac{\lambda}{4}\left(f_{\hat{x}}^{n+1} + f_{\hat{x}}^n\right) - \frac{\lambda}{4}\left(\bar{g}_{\hat{x}}^{n+1} + \bar{g}_{\hat{x}}^n\right)$$

where $\overline{g}^{*}(0, -\frac{4}{3R\Delta x}(\frac{m}{\rho})^2_{\hat{x}} - \frac{4}{3R\Delta x}(\frac{m}{\rho})(\frac{m}{\rho})_{x\overline{x}}, -\frac{4}{3R\Delta x}(\frac{m}{\rho})_{x\overline{x}}$.

The iteration method is defined by

$$w^o = v^n$$

$$v^{n+1} = w^p$$

$$w^{s+1} = v^n - \frac{\lambda}{4}((f(w^s))_{\hat{x}} + (f(v^n))_{\hat{x}})$$

$$- \frac{\lambda}{4}(\overline{g}(w^s)_{\hat{x}} + \overline{g}(v^n)_{\hat{x}}).$$

This merely amounts to an approximate solution of the implicit difference equations by successive substitutions where p is the number of iterations. We use the same value of p at all time steps, i.e. for all values of n. The truncation error of this method is $O(\Delta x^3)$ provided $p \geqq 2$. Since the implicit difference equations are unconditionally stable, we might conjecture that this iteration method has a favorable stability criterion. Unfortunately this is not the case. We will show that the method is unconditionally unstable if $\rho = 1, 2, 5, 6, 9, 10, \ldots$ and conditionally stable otherwise with the condition that λ be less than twice the bound that Courant, Friedrichs, and Lewy discovered [3].

3. Description of the Computed Solutions.

Professor H. Keller suggested the use of this problem. Ludloff and Filler have computed the solution to this problem using difference schemes of first order accuracy [6].

We compute the solution between $x = 0$ and $x = c$ at the points (t_n, x_i) where $x_i = i\Delta x$, $t_n = n\Delta t$, $i = 1, 2, \ldots M$, $n = 1, 2, \ldots N$, $M\Delta x = c > 1$. We assign the initial values of u as follows $(0 < d,\ d + 1 < c)$

$$u(0,x) = \left\{\begin{array}{ll} \pm 1 & 0 \leqq x \leqq d \\ \phi(x) \text{ (arbitrary)} & d < x < d + 1 \\ 0 & d + 1 \leqq x \leqq c \end{array}\right\} .$$

We choose $\phi(x)$ such that $u(o,x)$ is continuous and we usually require that $\partial u/\partial x$ also be continuous. We choose the initial values of p and ρ such that the solution is a forward facing simple wave if $R^{-1} = 0$, that is

$$\rho(0,x) = \rho_o \left[1 + \frac{\gamma-1}{2}\frac{u}{c_o}\right]^{\frac{2}{\gamma-1}}$$

$$p(0,x) = \frac{p_o}{\rho_o}\, \rho(0,x)^{\gamma} .$$

We fix the boundary conditions by requiring that $u(t_n,0) = u(0,0)$ and $u(t_n,c) = u(0,c)$ for all n. We must choose c large enough so that c remains downstream from the compression or rarefaction wave. These boundary conditions are somewhat dubious if $R^{-1} \neq 0$ since the Navier-Stokes equations are not hyperbolic, but the results appear reasonable. If $R^{-1} = 0$ and $u(0,0) = 1$ $(u(0,0) = -1)$, then we have a compression (rarefaction) wave. In the former case a shock will ultimately appear in the flow. In the latter case we can compute the exact solution $(R^{-1} = 0$, thus we have a simple wave) by solving a simple relation involving $\phi(x)$ [4]. Thus we have

a check on the accuracy of our finite difference methods. The results of the computations are described below. The computations were done on the IBM 7090 at the New York University Computing Center.

4. An Analysis of Stability.

By using the method of von Neumann we shall derive a stability criterion for the iteration method for the case where $R^{-1} = 0$. To use this method we must assume that $f(v) = Av$ where A is a constant matrix (A has real distinct eigenvalues in our case) [5]. Then the iteration method is given by

$$w^o = v^n$$

$$v^{n+1} = w^p$$

$w^{s+1} = v^n - \frac{\lambda}{4}(Aw^s_{\hat{x}} + Av^n_{\hat{x}}) = v^n + Nv^n + Nw^s$ where Nv denotes the operator $-\frac{\lambda}{4} Av_{\hat{x}}$. We see that

$$v^{n+1} = w^p = (I + 2N + 2N^2 + \cdots + 2N^p)v^n .$$

To use the method of von Neumann we assume that $v^n_i = K^n \exp(jwx_i)$ where $j = \sqrt{-1}$. Then we must evaluate the eigenvalues of the amplification matrix M where $v^{n+1} = Mv^n$. If $\mu = -(\lambda/2)\sin w\Delta x$, then $Nv^n = j\mu Av^n$. Therefore $M = I + 2\mu jA + \cdots + 2j^p\mu^p A^p$. If we let m and a denote the eigenvalues of M and A, and let $b = \mu a$, then

$$m = -1 + 2(1 + jb + \cdots + j^p b^p),$$

$$\text{or} \qquad m = -1 + 2\left[\frac{1 - j^{p+1}b^{p+1}}{1 - jb}\right] .$$

Therefore we have

$$|m|^2 = 1 + (-1)^n 4b^{2(n+1)} \left[\frac{1 + (-1)^n b^{2(n+1)}}{1 + b^2}\right] \quad \text{if} \quad p = 2n + 1,$$

and

$$|m|^2 = 1 + (-1)^{n+1} 4b^{2(n+1)} \left[\frac{1 + (-1)^{n+1} b^{2n}}{1 + b^2}\right] \quad \text{if} \quad p = 2n.$$

From these equations we see that $|m| > 1$ if $p = 1, 2, 5, 6, 9, 10, \ldots.,$ or if $|b| > 1$. If $p = 3, 4, 7, 8, \ldots.$ and $|b| \leq 1$, then $|m| \leqq 1$ which shows that the method is stable in this case and unstable in the preceding case. Note that the Courant, Friedrichs, Lewy condition is $|b| \leqq \frac{1}{2}$.

Computations were performed using the iteration method as described above in section 3. In one run we had $\phi(x) = (x-d)^2(2d + 3 - 2x) - 1$ and therefore the solution was a rarefaction wave ($R^{-1} = 0$ in all the tests of stability for the iteration method). With $b = 0.95$, $\Delta x = 0.025$, and using $p = 2$ instability was evident after 39 time steps, at which time the pressure took on negative values. We also used

$$\phi(x) = (x-d)^2(2x - 2d - 3) + 1 \tag{4.1}$$
$$+ (1/5)(x-d)(d + 1 - x)\sin([(1/4)\Delta x]\pi(x-d))$$

($[a]$ = integer part of a) which produced a compression wave with a superimposed oscillation. With $b = 0.95$, $\Delta x = 0.025$ and $p = 2$ instability occurred at 10 time steps and with $p = 5$ at 14 time steps. With $p = 3, 4$ or 7

both problems were run to 100 time-steps with no sign of instability. In the case of the compression wave a shock eventually formed in the flow. This did not cause instability.

One run was made where the solution was a rarefaction wave with $b = 0.45$, $\Delta x = 0.05$, and $p = 2$. No instability was evident after 200 time steps. This can probably be explained by inspection of the eigenvalues m of the amplification matrix M defined above. If $p = 2$, then $|m| = 1 + 4b^4$. Since $\log(1 + 4\cdot 0.95) = 0.63$ and $\log(1 + 4\cdot 0.45) = 0.064$, we see that approximately ten times as many cycles will be required to produce instability when $b = 0.45$. Note that the amplification after N time steps is governed by m^N, i.e. $(1 + 4b^4)^N$.

When $p = 2$ the iteration method is very similar to the Lax-Wendroff method. For $p = 2$ and $R^{-1} = 0$ the equations for the iteration method are (assume that $f(v) = Av$, where A is constant)

$$(4.2) \qquad v_i^{n+1} = v_i^n - \frac{\lambda}{2} A(v_{i+1}^n - v_{i-1}^n) + \frac{\lambda^2}{8} A^2(v_{i+2}^n - 2v_i^n + v_{i-2}^n)$$

and the Lax-Wendroff equations are

$$(4.3) \qquad v_i^{n+1} = v_i^n - \frac{\lambda}{2} A(v_{i+1}^n - v_{i-1}^n) + \frac{\lambda^2}{2} A^2(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

The Lax-Wendroff equations are derived from equations (2.1) by replacing derivatives by difference quotients. If we replaced $\dfrac{\partial^2 v}{\partial x^2}$ by $\dfrac{v_{i+2}^n - 2v_i^n + v_{i-2}^n}{4(\Delta x)^2}$ instead of $(1/\Delta x)^2\, v_{x\bar{x}}^n$,

then the Lax-Wendroff method would yield equations (4.2) instead of (4.3). In this case the Lax-Wendroff method would produce an unstable difference scheme.

We are unable to compute a stability criterion for the Lax-Wendroff method as applied to the Navier-Stokes equations. We can obtain an empirical stability criterion by computing the solution for a compression wave, for various values of R, Δx and Δt. We compute the solution for 100 time-steps and observe the value of Δt for which the computation becomes unstable at fixed values of Δx and R. In these computations we use the Lax-Wendroff method on the equation (2.2). The initial conditions are determined by $\phi(x)$ as given in equation (4.1). We assume the scheme is stable for a given Δt if there is no obvious instability after 100 time steps. The results of the computations are shown in table 1. The largest value of Δt for which the calculation is stable is given along with the minimum value of Δt for which the calculation is unstable. The simple explicit scheme for the heat equation $\partial u/\partial t = \sigma\partial^2 u/\partial x^2$ has the stability criterion $\Delta t = (1/2\sigma)(\Delta x)^2$. The derivative of second order in the Navier-Stokes equations (2.2) has the coefficient $4/3R\rho$. If the Navier-Stokes equations behave as a parabolic system we might therefore expect the stability criterion $\Delta t = (3R\rho/8)(\Delta x)^2$. If the behavior is hyperbolic we expect the criterion $\Delta t = (1/(u+c))\Delta x$ where the maximum value of $u + c$ at $t = 0$ is used. From table 1 we see that the behavior is parabolic, that is $\Delta t \sim (\Delta x)^2$, for $R = 1$ and $R = 10$. It is hyperbolic, $\Delta t \sim \Delta x$ for $R = 10^4$.

As far as these results are concerned we see that the following empirical stability criterion will suffice

$$\Delta t = \min \left[(\Delta x/(u+c)),\ (1/2)(3R\rho/8)(\Delta x)^2\right] \quad .$$

We are forced to halve the usual parabolic condition. In table 1 the range of Δt in which instability occurs for various Δx and R is shown along with the values of Δt given by the various stability criteria.

The iteration method was also used to compute solutions to the Navier-Stokes equations. The values obtained for the pressure using this method agreed with the values obtained using the Lax-Wendroff method (applied to equations (2.2)), to within $0.3^{\circ}/\circ$ for $\Delta x = 0.05$, $R = 10$ at 40 time steps. This agreement seems to be consistent with the accuracy of these methods as discussed in the following section.

In figure 1 the values of velocity obtained using the Lax-Wendroff method at $R^{-1} = 0.1$ and $R^{-1} = 0$ are plotted. Both values are at time $t = 0.56$. All initial conditions are the same for both calculations.

5. The Accuracy of These Finite Difference Methods.

The truncation error of both the iteration method and the Lax-Wendroff method is $O(\Delta x^3)$. We checked this only for $R^{-1} = 0$; it is undoubtedly true for $R^{-1} \neq 0$ as well. If we take $R^{-1} = 0$ and choose $\phi(x)$ such that the solution is a rarefaction wave, then we can compute the exact solution by solving an equation involving $\phi(x)$. We compare this

solution with that obtained by the finite difference schemes. This is done in tables A, B and C below. These tables give the maximum percentage error in the pressure. The error in the Lax-Wendroff method is not greatly different from that in the iteration method.

The error is certainly not $O(\Delta x^3)$. The error might be considerably reduced if the exact solution were analytic, at least smoother than the ones used here. This is determined by the choice of $u(0,x)$. In the case of tables A and B the second derivative of the exact solution is not continuous since $\partial\phi^2/\partial x^2 \neq 0$ at $x = d$ and $x = 1 + d$ when $t = 0$. That is $u \in c^1$ but $u \notin c^2$. In table C $u \in c^2$ but $u \notin c^3$. These difference methods have been used on a problem for which the solution is analytic [7]. In this case the error was $O(\Delta x^3)$.

Table A. The maximum percentage error in the values of pressure obtained by the iteration method when $u(t,x) \in c^1$.

Δx	Time-steps 20	40	100
0.1	2.1	1.9	1.8
0.05	0.66	0.82	0.90
0.01	0.034	0.056	0.099

Table B. The maximum percentage error in the values of pressure obtained by the Lax-Wendroff method when $u(t,x) \in c^1$.

Δx	Time-steps 20	40	100
0.1	1.34	1.4	1.35
0.05	0.48	0.65	0.76
0.01	0.024	0.041	0.071

Table C. The maximum percentage error in the values of pressure obtained by the Lax-Wendroff method when $v(t,x) \varepsilon c^2$.

Δx	Time-steps 20	40	100
0.1	1.2	1.33	1.31
0.05	0.36	0.52	0.65
0.01	0.006	0.015	0.04

The iteration method, using three iterations per time-step, requires 240 seconds to compute 100 time-steps when 300 mesh points are used (an IBM 7090 is used for these calculations). This is 2.7×10^{-3} seconds per iteration per mesh point. The Lax-Wendroff method also requires 240 seconds to compute 100 time-steps when 300 mesh points are used. Note that both these calculations are for the Navier-Stokes equations which require more time than the equations of inviscid flow.

6. Computation for Flows Which Contain a Shock.

If the flow contains a shock these finite difference methods can still be used to compute an approximate solution [1,2]. The different methods yield results which differ considerably. The Lax-Wendroff method yields much better results when the

equations of motion are written in conservation form as Professor Lax has indicated [1,2]. The iteration method fails even though the equations are in conservation form.

These results are obtained by computing the solution for a compression wave $(R^{-1} = 0)$. Enough time steps are used so that the shock created by the compression wave reaches an apparent steady state. All of the methods seem to be stable provided the shock strength is not too great. In figure 2 the values of u computed by the iteration method are compared with those computed by the Lax-Wendroff method (equations in conservation form) for a shock of strength $((P_1 - P_o)/\rho_o) = 0.40$. Clearly the iteration method is not acceptable even for this weak shock.

In figure 2 the values of u obtained by the Lax-Wendroff method (conservation form) are shown. In table 2 the state behind the shock is given as calculated by the Lax-Wendroff method, both for the equations in conservation form and in the form of (2.2). Both the minimum values and the values at the peak immediately behind the shock are given (see figure 2).

For weak shocks we can obtain an approximate value for the state immediately behind the shock by assuming that the flow behind the shock is isentropic. Then we may assume that the characteristics behind the shock are straight lines. Thus we can determine the value of $u + c$ behind the shock. The state behind the shock then follows from the Hugoniot relations. This approximate solution is given in the tables. The values for the constant state behind the compression wave are also

given. These are the values used to determine the left-hand boundary condition.

If we know the pressure jump across a shock the velocity and density can be obtained from the Hugoniot relations. Thus we can use the pressure calculated by the difference scheme to obtain a check on the values of u and ρ. The values of u and ρ computed from the Hugoniot relations are shown in the table.

We see from table 2 that the values computed with the equations in conservation form more nearly satisfy the Hugoniot relations. Also, the method based on equations (2.2) is unstable at a lower shock strength than the method based on the equations in conservation form. The latter method is unstable when the shock strength equals 13, and perhaps for shocks of lower strength although it seems to be stable when the shock strength equals 6.

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Δx	$10^2 \Delta t$ (stable)	$10^2 \Delta t$ (unstable)	$10^2 (\Delta x / u+c)$	$10^2 (^1/2)(3R\rho/8)\Delta x^2$
0.05	0.056	0.061	2.1	0.047
0.025	0.014	0.015	1.0	0.012
0.01	0.0022	0.0024	0.42	0.0019
0.05	0.56	0.61	2.1	0.47
0.025	0.14	0.15	1.0	0.12
0.01	0.022	0.024	0.42	0.019
0.05	2.0	2.1	2.1	4.7
0.025	1.0	1.1	1.0	1.2
0.01	0.22	0.24	0.42	0.19
0.05	2.1	2.3	2.1	470.
0.025	1.0	1.1	1.0	120.
0.01	0.40	0.43	0.42	19.

Table 1

th)/ρ_o		Const.state behind the shock	Approx.solu. (isentropic)	Calculated solution min.	Calculated solution max.	From Hugoniot relations min.	From Hugoniot relations max.	
	ρ	1.49	1.48	1.48	1.51	1.48	1.52	Method based on equations in conser-vation form
	p	7.20	7.17	7.15	7.48	--	--	
	u	1.00	0.995	0.99	1.07	.99	1.08	
	ρ	2.16	2.04	2.06	2.11	2.06	2.12	
	p	3.02	2.92	2.99	3.10	--	--	" "
	u	1.00	0.984	1.00	1.04	1.00	1.04	
	ρ	4.21	3.13	3.20	3.27	3.31	3.33	
	p	1.93	1.58	1.80	1.83	--	--	" "
	u	1.00	0.955	1.04	1.05	1.04	1.05	
	ρ	1.49	1.48	1.47	1.50	1.46	1.51	Method based on equations (2.2)
	p	7.20	7.17	7.04	7.37	--	--	
	u	1.00	0.995	1.03	1.13	0.96	1.05	
	ρ	2.16	2.04	2.02	2.13	1.96	2.03	
	p	3.02	2.92	2.74	2.92	--	--	" "
	u	1.00	0.984	1.09	1.15	0.92	0.97	
	ρ	4.21	3.13					
	p	1.93	1.58	unstable				" "
	u	1.00	0.955					

Table 2. The state immediately behind the shock as calculated by the Lax-Wendroff method.

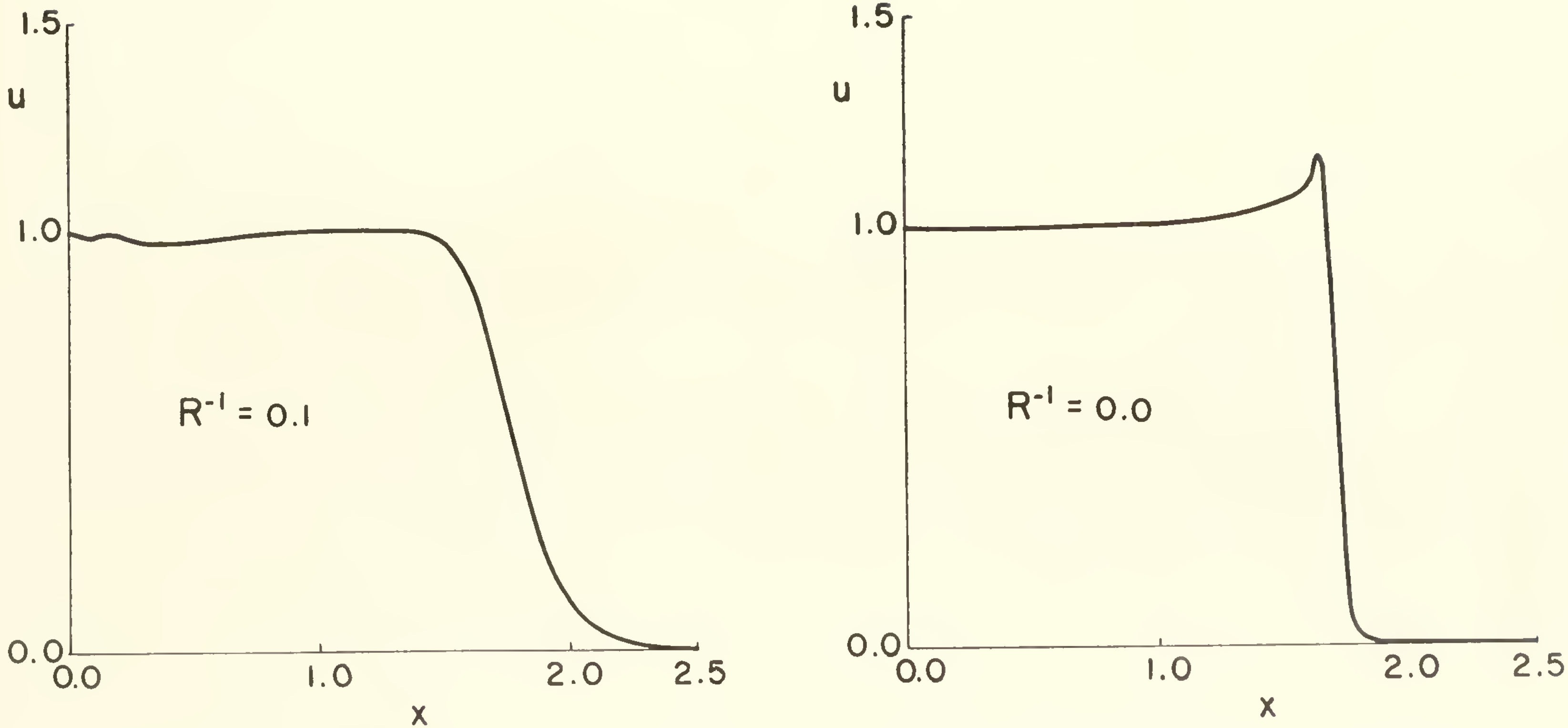

Figure 1: VELOCITY AS COMPUTED BY THE LAX-WENDROFF METHOD WHEN $\Delta x = 0.05$, $t = 0.56$.

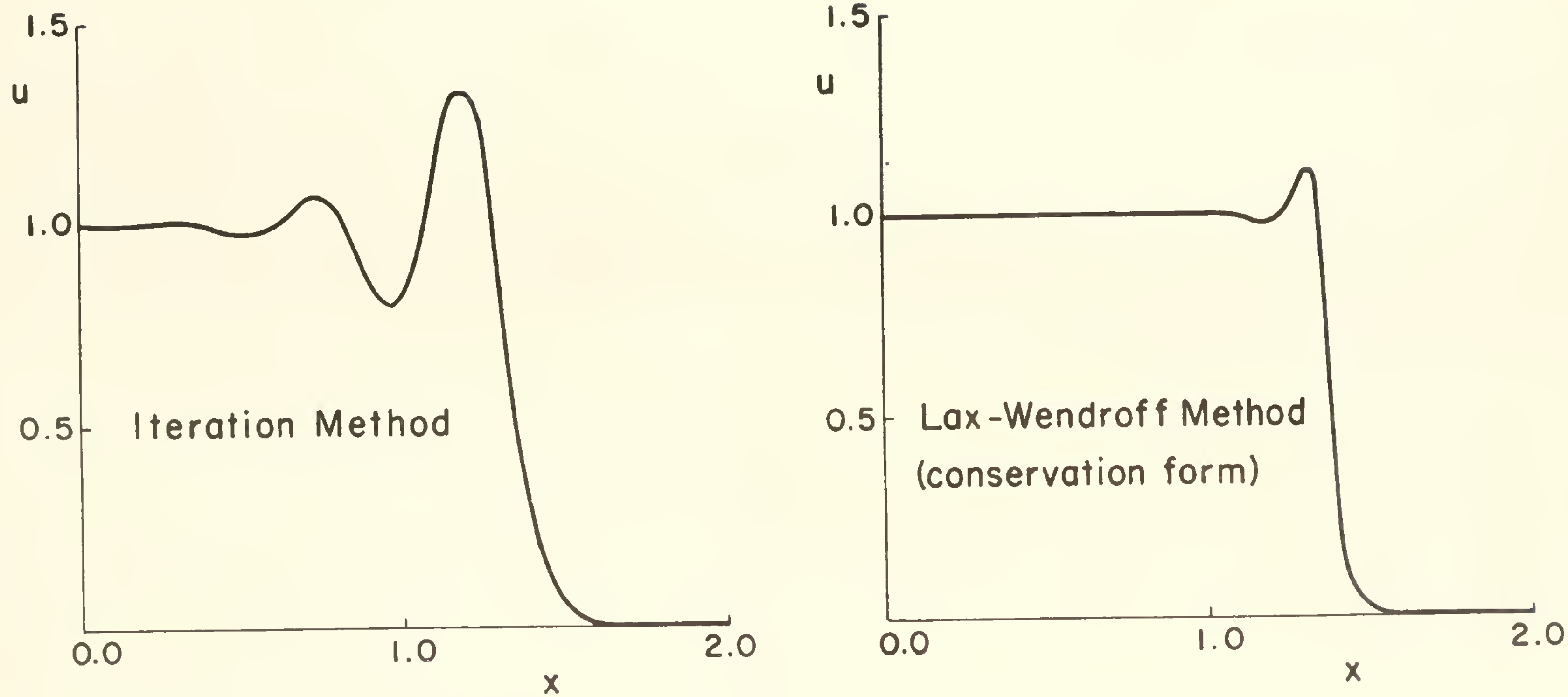

Figure 2. VELOCITY AS COMPUTED BY TWO METHODS WHEN $\Delta x = 0.05$, $t=1.12$, SHOCK STRENGTH = 0.41, $R^{-1} = 0.0$.